Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-04-17

Conceptual Review

Question 1. Is there a vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\nabla \times \mathbf{F} = \langle x, 0, 0\rangle$?

similar: does there exist f s.t. $\nabla f = \langle y, -x\rangle$? ($P = y$, $Q = -x$)

$Q_x - P_y = -2 \neq 0$

i.e. I computed the curl: $\langle 0, 0, -2\rangle \neq \vec{0}$

so NO.

$\nabla \cdot (\nabla \times \vec{F}) = 0$. But $\nabla \cdot \langle x, 0, 0\rangle = 1$. So no.

for any "reasonable" $\vec{F}$

Problems

Problem 1. Let D be the sphere $x^2 + y^2 + z^2 = 9$, oriented positively (outwards).

(a) Without parametrizing, compute the unit normal $\mathbf{n}$ for D in terms of x, y, z. Hint: You did this a lot in Chapter 14.

A normal vec is given by the gradient $\langle 2x, 2y, 2z\rangle$. Rescale: $\dfrac{\langle 2x, 2y, 2z\rangle}{2\sqrt{x^2+y^2+z^2}} = \frac{1}{3}\langle x, y, z\rangle$

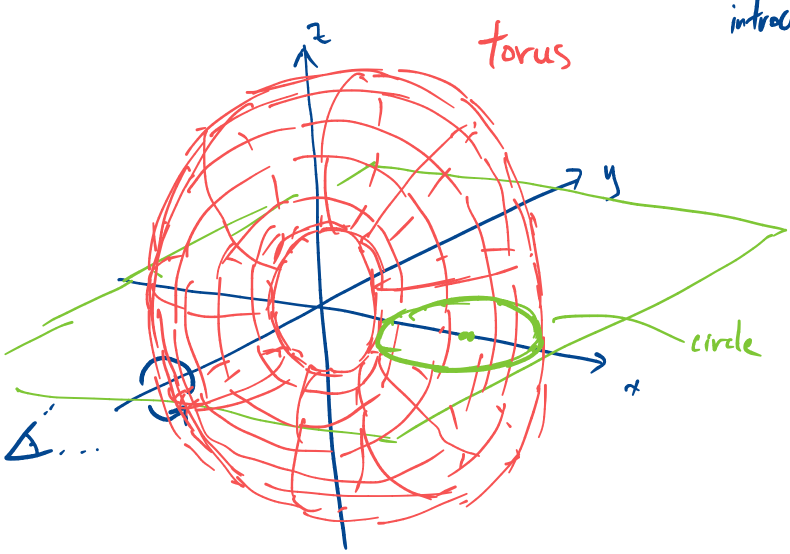
(b) Compute the flux of the vector field $\mathbf{F} = \langle x, y, z\rangle$ through D. Hint: The surface area of a sphere of radius R is $4\pi R^2$.

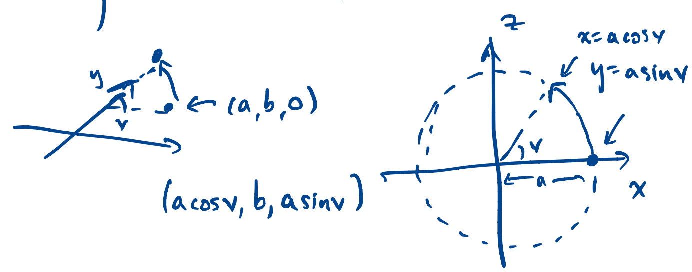
$$\iint_D \vec{F}\cdot\vec{n}\, dS = \iint_D \langle x,y,z\rangle \cdot \tfrac{1}{3}\langle x,y,z\rangle\, dS = \iint_D \tfrac{1}{3}(x^2+y^2+z^2)\, dS = 3\iint_D 1\, dS$$

$$= 3(4\pi 3^2) = 108\pi$$

Problem 2. The curve $\langle 2 + \cos u, \sin u\rangle$, $0 \le u \le 2\pi$ in the xy-plane is rotated around the y-axis. Parametrize the resulting surface, and describe it.

introduce another parameter $0 \le v \le 2\pi$ for the rotation.

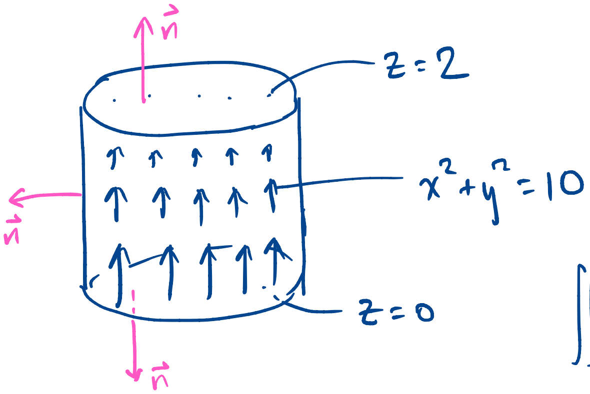




$$\langle (2+\cos u)\cos v,\ \sin u,\ (2+\cos u)\sin v\rangle$$

Check in CalcPlot3D!

Problem 3. Compute the flux of the vector field $\langle 0, 0, 4 - z^2 \rangle$ outwards through the closed cylinder with lateral side $x^2 + y^2 = 10$ and lids $z = 0$ and $z = 2$. Note that this surface has three parts.



- No flux through lateral side (because $\vec{F} \cdot \vec{n} = 0$ there)
- No flux through top (because $\vec{F} = \vec{0}$ there, so $\vec{F} \cdot \vec{n} = 0$ too)

$$\iint_D \vec{F} \cdot \vec{n} \, dS = \iint_D \langle 0, 0, 4 - z^2 \rangle \cdot \langle 0, 0, -1 \rangle \, dx\,dy$$

$$= \iint_D -4 \, dx\,dy = -4 \iint_D 1 \, dx\,dy = \boxed{-40\pi}$$

Problem 4. Compute the divergence of the vector field in Problem 1. Integrate it over the 3d interior of the sphere: $x^2 + y^2 + z^2 \leq 9$. What do you get?

Compute the divergence of the vector field in Problem 3. Integrate it over the 3d interior of the cylinder. What do you get?

Will come back to this when we cover the divergence theorem.